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G-FUNCTIONS

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$$g(z) = \sum_{n=0}^{\infty} g_n z^n$$

where the g_n satisfy the following conditions,

- The g_n all belong to an algebraic numberfield k.
- $g(z)$ satisfies a linear differential equation with polynomial coefficients.
- $$\sum_v \max(|g_0|_v, |g_1|_v, \ldots, |g_n|_v) = O(n)$$

  as $n \to \infty$, where the sum is over all normalised valuations $|.|_v$ of k.

EXAMPLES of G-FUNCTIONS

- Functions, algebraic over $k(z)$.
- Polylogarithms

$$L_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}.$$

- Hypergeometric functions

$$F(a,b,c|z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n$$

where $(x)_n = x(x+1)\cdots(x+n-1)$ and $a, b, c \in \mathbb{Q}$.

QUESTION

What are G-functions?

(EXAMPLE) Consider

$$z(z^2-1)f'' + (3z^2-1)f' + (z+B)f = 0 \tag{1.3}$$

where B is an arbitrary rational number. For any solution

$$g(z) = \sum_{n=0}^{\infty} g_n z^n$$

we have the recurrence relation

$$n^2 f_n = Bf_{n-1} + (n-1)^2 f_{n-2} \qquad f_0 = 1,\ f_1 = B.$$

When $B = 0$, we have $f_{2n+1} = 0, f_{2n} = \binom{2n}{n}^2/16^n$. Are there any other B for which $g(z)$ is a G-function?

A conjecture of E.Bombieri and B.Dwork states that differential equations having G-function solutions 'arise' from Gauss-Manin systems corresponding to 1-parameter families of algebraic varieties.

QUESTION

Let $g(z)$ be a non-algebraic G-function, let $\xi \in \overline{\mathbb{Q}}$; $\xi \neq 0$ and let $|.|_v$ be a valuation such that $g(\xi)$ converges v-adically. Do we have $g(\xi) \in \overline{\mathbb{Q}}$ or not?

(THEOREM). Let j be the modular j-invariant of an elliptic curve with complex multiplication and endomorphism algebra $\mathbb{Q}(\sqrt{-1})$. Suppose that $\xi = 1 - 12^3/j$ has complex absolute value less than 1. Then,

$$F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}|\xi\right) \in \overline{\mathbb{Q}}.$$

(EXAMPLE). Take $j = j(2i)$. Then $1 - 12^3/j = 1323/1331$ and

$$F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}|\frac{1323}{1331}\right) = \frac{3}{4}\sqrt[4]{11}.$$

(OBSERVATION). We have 7-adically,

$$F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}|\frac{1323}{1331}\right)_7 = \frac{1}{4}\sqrt[4]{11}.$$

PICARD-FUCHS EQUATIONS

Let K be an algebraic numberfield and consider the family of elliptic curves $\mathcal{E}_t$ given by

$$y^2 = x^3 + A(t)x + B(t), \qquad A(t), B(t) \in K[t].$$

Assume $\Delta(0) \neq 0$, where $\Delta(t) = \Delta(\mathcal{E}_t) = 4A(t)^3 + 27B(t)^2$ and assume that $j(t) = j(\mathcal{E}_t)$ is non-constant.

$$\Omega(t) = \frac{dx}{2y}, \qquad N(t) = x\frac{dx}{2y} + \gamma\frac{dx}{2y}$$

on $\mathcal{E}_t$, where $\gamma \in K$ is to be specified later if necessary.

There exists $\mathcal{A} \in M_2(K(t))$ such that

$$\frac{d}{dt}\begin{pmatrix}\Omega(t)\\ N(t)\end{pmatrix} \equiv \mathcal{A}\begin{pmatrix}\Omega(t)\\ N(t)\end{pmatrix} \text{ (Modulo exact forms).}$$

(PICARD-FUCHS EQUATION)

$$\frac{d}{dt}\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \mathcal{A}\begin{pmatrix}y_1\\ y_2\end{pmatrix}$$

(FUNDAMENTAL SOLUTION MATRIX) $M(t) \in M_2(K[[t]])$ such that

$$\frac{d}{dt}M(t) = \mathcal{A}(t)M(t) \text{ and } M(0) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

(PROPERTIES)

- $\det(M(t)) = 1$
- The elements of $M(t)$ are G-functions.
- The elements of $M(t)$ have transcendence degree 3 over $\mathbb{C}(t)$.

Let $\mathcal{E}_t$: $y^2 = x^3 - x - t$ and $\Omega(t) = dx/2y$, $N(t) = xdx/2y$. Then $\Delta(t) = 27t^2 - 4$,

$$\mathcal{A}(t) = \frac{1}{2(27t^2-4)}\begin{pmatrix}-9t & -6\\ 2 & 9t\end{pmatrix}$$

and

$$M(t) = \begin{pmatrix} F(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}|\frac{27}{4}t^2) & \frac{3}{4}tF(\frac{7}{12}, \frac{11}{12}, \frac{3}{2}|\frac{27}{4}t^2) \\ -\frac{1}{4}tF(\frac{5}{12}, \frac{13}{12}, \frac{3}{2}|\frac{27}{4}t^2) & F(-\frac{1}{12}, \frac{7}{12}, \frac{1}{2}|\frac{27}{4}t^2) \end{pmatrix}.$$

Let $\mathcal{E}_t$: $y^2 = x^3 + tx + 1$ and $\Omega(t) = dx/2y$, $N(t) = xdx/2y$. Then $\Delta(t) = 4t^3 + 27$,

$$\mathcal{A}(t) = \frac{1}{2(4t^3+27)}\begin{pmatrix}-2t^2 & 9\\ 3t & 2t^2\end{pmatrix}$$

and

$$M(t) = \begin{pmatrix} F(\frac{1}{12}, \frac{7}{12}, \frac{2}{3}|-\frac{4}{27}t^3) & \frac{1}{6}tF(\frac{5}{12}, \frac{11}{12}, \frac{4}{3}|-\frac{4}{27}t^3) \\ \frac{1}{36}t^2F(\frac{7}{12}, \frac{13}{12}, \frac{5}{3}|-\frac{4}{27}t^3) & F(-\frac{1}{12}, \frac{5}{12}, \frac{1}{3}|-\frac{4}{27}t^3) \end{pmatrix}.$$

(THEOREM) Suppose E_0 is isogenous over K to E_a via $\phi:\ E_0 \to E_a$. Let $N = \deg\phi$. Let $\alpha, \beta \in K$ be defined by

$$\phi^*\omega_a = \alpha\omega_0 \qquad \phi^*\eta_a = \frac{N}{\alpha}\eta_0 + \beta\omega_0.$$

Let v be a valuation of K such that $|a|_v < \max(1, \rho_v)$. When v is finite, also suppose that $|A(0)|_v, |B(0)|_v \leq 1$ and $|6\Delta(0)|_v = |-2A(0)B'(0) + 3B(0)A'(0)|_v = 1$. Then,

$$\mathrm{Tr}\left(\begin{pmatrix} N/\alpha & 0 \\ -\beta & \alpha \end{pmatrix} M(a)_v\right) \in \mathbb{Z}$$

where $M(a)_v$ denotes the v-adic evaluation of $M(a)$. Moreover, when v is finite, the trace on the left hand side of (4.1) is $\leq 2\sqrt{N}$.

(COROLLARY) Suppose in addition to the above assumptions that E_0 is a C.M.curve. Choose γ such that $\mathrm{End}(E_0)$ acts diagonally on ω_0, η_0. Let $-d = \mathrm{discr}(\mathrm{End}(E_0))$ and write

$$M(t) = \begin{pmatrix} m_{11}(t) & m_{12}(t) \\ m_{21}(t) & m_{22}(t) \end{pmatrix}.$$

Then,

$$\frac{N}{\alpha}m_{11}(a)_v,\ \alpha m_{22}(a)_v - \beta m_{12}(a)_v \in \frac{1}{\sqrt{-d}}\mathbb{Z}\left[\frac{d+\sqrt{-d}}{2}\right].$$

(EXAMPLE 1) Take $a = 11^{3/2}/14$. Then $j(E_a) = 66^3$ and there is an isogeny of degree 2 from E_0 to E_a. Let

$$f(z) = 2 \cdot 11^{-1/4} F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}|z\right)$$

Then

$$f\left(\frac{1323}{1331}\right)_\infty = \frac{3}{2} \qquad f\left(\frac{1323}{1331}\right)_7 = \frac{1}{2}.$$

Moreover,

$$\frac{21}{242}F\left(\frac{7}{12}, \frac{11}{12}, \frac{3}{2}|\frac{1323}{1331}\right) + F\left(-\frac{1}{12}, \frac{7}{12}, \frac{1}{2}|\frac{1323}{1331}\right) = \frac{3}{2}\cdot 11^{-1/4}$$

$$\frac{21}{242}F\left(\frac{7}{12}, \frac{11}{12}, \frac{3}{2}|\frac{1323}{1331}\right)_7 + F\left(-\frac{1}{12}, \frac{7}{12}, \frac{1}{2}|\frac{1323}{1331}\right)_7 = \frac{1}{2}\cdot 11^{-1/4}.$$

(EXAMPLE 2) Take a such that $j(E_a) = (3(724 + 513\sqrt{2}))^3$. Then E_a is isogenous to E_0 of degree 4. Let

$$f(z) = 4 \cdot (91 + 60\sqrt{2})^{-1/4} F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}|z\right)$$

and

$$\xi = 3^3 \cdot 7^2 \cdot 11^2(3-2\sqrt{2})(5+\sqrt{2})^3(7-\sqrt{2})^3/(23\cdot 47)^3.$$

Then

$$f(\xi)_\infty = \frac{5}{2} \qquad f(\xi)_{3+\sqrt{2}} = \frac{3}{2}$$
$$f(\xi)_{3-\sqrt{2}} = \frac{1+i}{2} \qquad f(\xi)_{11} = \frac{1+2i}{2}.$$

(THEOREM) Suppose E_0 is a C.M.curve and suppose γ is chosen such that the action of $\mathrm{End}(E_0)$ on ω_0, η_0 is diagonal. Suppose E_0 is isogenous over K to E_a via $\phi:\ E_0 \to E_a$. Let $N = \deg\phi$. Let $\alpha, \beta \in K$ be defined by

$$\phi^*\omega_a = \alpha\omega_0 \qquad \phi^*\eta_a = \frac{N}{\alpha}\eta_0 + \beta\omega_0.$$

Let v be a finite place of K which satifies $|A(0)|_v, |B(0)|_v \leq 1$ and

$$|6\Delta(0)|_v = |-2A(0)B'(0) + 3B(0)A'(0)|_v = 1.$$

Suppose that $\overline{E_0}$ is ordinary and $|a|_v < 1$. Let $-d$ =discr(End($\overline{E_0}$)). Then there exists $\epsilon \in \mathbb{Z}[(d+\sqrt{-d})/2]$ with $\epsilon\bar{\epsilon} = N$ such that

$$M(a)_v = \frac{1}{N}\begin{pmatrix}\alpha & 0\\ \beta & N/\alpha\end{pmatrix}\begin{pmatrix}\bar{\epsilon} & 0\\ 0 & \epsilon\end{pmatrix}.$$

(EXAMPLE) Let a be such that $j(E_a) = (6(2927+1323\sqrt{5}))^3$. Then E_0 is 5-isogenous to E_a. Moreover, $\overline{E_0}$ is ordinary. Let

$$\xi = 2\cdot 3^3 \cdot\cdot 7^2\sqrt{5}(47-20\sqrt{5})^2(673+357\sqrt{5})^3/(11\cdot 59\cdot 71)^3.$$

Then

$$F\left(\frac{1}{12},\frac{5}{12},\frac{1}{2}|\xi\right)_5 = -\frac{1}{2+i}(161+120\sqrt{5})^{1/4}$$
$$F\left(\frac{7}{12},\frac{11}{12},\frac{1}{2}|\xi\right)_5 = 0$$
$$F\left(\frac{5}{12},\frac{13}{12},\frac{3}{2}|\xi\right)_5 = -\frac{(1229-515\sqrt{5})(161+120\sqrt{5})^{1/4}}{7(47-20\sqrt{5})(2+i)}$$
$$F\left(-\frac{1}{12},\frac{7}{12},\frac{1}{2}|\xi\right)_5 = -(2+i)(161+120\sqrt{5})^{-1/4}.$$

(THEOREM) Suppose E_0 is C.M.curve and γ is chosen such that the action of $\mathrm{End}(E_0)$ on ω_0, η_0 is diagonal. Let $E_a,\ a \in K$ be another C.M.curve and suppose that there exists a valuation v on K such that $|a|_v < \max(1, \rho_v)$. When v is finite we also assume that $|A(0)|_v, |B(0)|_v \leq 1$ and $|6\Delta(0)|_v = |-2A(0)B'(0) + 3B(0)A'(0)|_v = 1$. Let δ be such that the action of $\mathrm{End}(E_a)$ on $\omega_a, \eta_a + \delta\omega_a$ is diagonal. Then,

$$\frac{m_{11}(a)_v(m_{22}(a)_v + \delta m_{12}(a)_v)}{m_{12}(a)_v(m_{21}(a)_v + \delta m_{11}(a)_v)} \in \mathbb{Q}(\mathrm{End}(E_0), \mathrm{End}(E_a)).$$

(EXAMPLE) Let a be such that $j(E_a) = 20^3$. Then $\mathrm{End}(E_a) = \mathbb{Z}[\sqrt{-2}]$ and $\delta = 1/\sqrt{30}$. Denote

$$f(z) = F(\tfrac{1}{12}, \tfrac{5}{12}, \tfrac{1}{2}|z) \qquad g(z) = F(\tfrac{7}{12}, \tfrac{11}{12}, \tfrac{3}{2}|z)$$
$$h(z) = F(\tfrac{5}{12}, \tfrac{13}{12}, \tfrac{3}{2}|z) \qquad k(z) = F(-\tfrac{1}{12}, \tfrac{7}{12}, \tfrac{1}{2}|z)$$

and

$$Q(z) = \frac{f(z)(2250k(z) + 105g(z))}{g(z)(-49h(z) + 105f(z))}.$$

Then

$$Q\left(\frac{98}{125}\right)_\infty = (\sqrt{2}+1)^4 \qquad Q\left(\frac{98}{125}\right)_7 = -\frac{(2\sqrt{2}-1)^2}{7}.$$